DAUGAVET CENTERS AND DIRECT SUMS OF BANACH SPACES

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Abstract. A linear continuous nonzero operator G: $X \to Y$ is a Daugavet center if every rank-1 operator T: $X \to Y$ satisfies $\|G + T\| = \|G\| + \|T\|$. We study the case when either X or Y is a sum $X_1 \oplus_F X_2$ of two Banach spaces X_1 and X_2 by some two-dimensional Banach space F. We completely describe the class of those F such that for some spaces X_1 and X_2 there exists a Daugavet center acting from $X_1 \oplus_F X_2$, and the class of those F such that for some pair of spaces X_1 and X_2 there is a Daugavet center acting into $X_1 \oplus_F X_2$. We also present several examples of such Daugavet centers.

1. Introduction

In the present paper we consider real Banach spaces which do not equal $\{0\}$, and denote them E, X or Y. A linear continuous nonzero operator G: $X \to Y$ is called a *Daugavet center* [3] if every rank-1 operator T: $X \to Y$ satisfies the equation

$$\|G + T\| = \|G\| + \|T\|. \tag{1.1}$$

Definition 1.1. *We say that X is a Daugavet domain if there exists a Daugavet center G: $X \to Y$ for some Y, and is a Daugavet range if there is a Daugavet center G: $E \to X$ for some E.*

Throughout this paper $F = (\mathbb{R}^2, \|\cdot\|)$ with $\|(1, 0)\| = \|(0, 1)\| = 1$ and

$$\|(a_1, a_2)\| = \|(|a_1|, |a_2|)\| \tag{1.2}$$

for every $(a_1, a_2) \in F$. For Banach spaces X_1 and X_2 their F-sum $X_1 \oplus_F X_2$ is the space of all pairs (x_1, x_2) where $x_1 \in X_1$ and $x_2 \in X_2$, $\|(x_1, x_2)\| := \|(\|x_1\|, \|x_2\|)\|_F$.

We introduce the following order on F: $(a_1, a_2) \geq (b_1, b_2)$ if $a_1 \geq b_1$ and $a_2 \geq b_2$. It follows from (1.2) and a convexity argument that for every $(a_1, a_2), (b_1, b_2) \in F$ with $(|a_1|, |a_2|) \leq (|b_1|, |b_2|)$ the inequality $\|(a_1, a_2)\| \leq \|(b_1, b_2)\|$ holds true. In this partial order F is a Banach lattice [8], so we will use the term "two-dimensional lattice" for F in the sequel.

The problem which we solve in this paper, consists of two parts: first, we characterize the class of those F for which there exist X_1 and X_2 such that

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$X_1 \oplus_F X_2$ is a Daugavet domain, and secondly, we characterize the class of those F for which there are X_1 and X_2 such that $X_1 \oplus_F X_2$ is a Daugavet range.

Remark that a Daugavet domain and a Daugavet range are generalizations of a Banach space with the Daugavet property, and this motivates our interest in the subject. A Banach space X is said to have the Daugavet property if the identity operator Id: $X \to X$ is a Daugavet center. The study of spaces with the Daugavet property is a rapidly developing branch of Banach space theory (see [6], [11], [13], and the most recent developments in [5], [7]). The following classical spaces have the Daugavet property: $C(K)$ where K is a compact without isolated points [4], $L_1(\mu)$ and $L_\infty(\mu)$ where μ has no atoms [9], and many Banach algebras ([14], [15]). Some exotic spaces have the Daugavet property as well, for instance, Talagrand's space ([6], [12]) and Bourgain-Rosenthal's space ([2], [7]).

Let us recall some recent results [3] related to Daugavet centers. If G is a Daugavet center then (1.1) also holds true when T is a strong Radon-Nikodým operator, e.g., a weakly compact operator. If X is a Daugavet domain or a Daugavet range then X contains subspaces isomorphic to ℓ_1, is non-reflexive and does not have an unconditional basis (countable or uncountable). One cannot even embed such an X into a space having an unconditional basis or having a representation as unconditional sum of reflexive subspaces. In [10] Popov proves that every isometric embedding of $L_1[0,1]$ into itself is a Daugavet center. However, in [3] one can find examples of Daugavet centers which are not isometries.

The present work is inspired by [1]. It was shown in [1] and [6] that if X_1 and X_2 have the Daugavet property and $F = \ell_1^{(2)}$ or $\ell_\infty^{(2)}$ then $X_1 \oplus_F X_2$ has the Daugavet property as well. In [1] the authors prove that $X_1 \oplus_F X_2$ has the Daugavet property only if $F = \ell_1^{(2)}$ or $\ell_\infty^{(2)}$. In our paper we generalize these results of [1], but use a new approach to the problem. Surprisingly in both parts of our problem we discover other spaces apart from $F = \ell_1^{(2)}$ and $F = \ell_\infty^{(2)}$, which satisfy our demands.

Our approach is based on a necessary condition for a general Banach space X to be a Daugavet domain and on a necessary condition for X to be a Daugavet range. We deduce these two conditions in Section 2 (see Definition 2.1, Lemma 2.5 and Definition 2.2, Lemma 2.6) and then we show in Section 3 how they depend on F when $X = X_1 \oplus_F X_2$ (see Lemma 3.6 and Lemma 3.7).

In Section 4 we find a rather small class $\mathfrak{N}_2$ such that if $X_1 \oplus_F X_2$ is a Daugavet range then $F \in \mathfrak{N}_2$, and in Section 5 we discover the analogous class $\mathfrak{M}_2$ for the case of a Daugavet domain. Then for every $F \in \mathfrak{M}_2$ we present an example of a Daugavet center acting *from* a sum of two Banach spaces by F (see Proposition 6.2), and this solves the first part of our problem. In a very similar way we solve its second part, namely we give examples of Daugavet centers acting *into* a sum of two Banach spaces by F

for every $F \in \mathfrak{N}_2$ (see Proposition 6.6). The obtained results illustrate that the notions of a Daugavet domain and a Daugavet range do not refer to the same Banach spaces.

Throughout this paper B_X denotes the closed unit ball of X and S_X denotes its unit sphere. We use the notation

$$B_F^+ := \{a \in B_F \colon a \geq 0\}$$

for the positive part of the unit ball and

$$S_F^+ := \{a \in S_F \colon a \geq 0\}$$

for the positive part of the unit sphere of F. We denote

$$S(B_X, z^*, \varepsilon) := \{x \in B_X \colon z^*(x) > 1 - \varepsilon\}$$

the slice of B_X determined by $z^* \in S_{X^*}$ and $\varepsilon > 0$.

$$S(B_{X^*}, z, \varepsilon) = \{x^* \in B_{X^*} \colon x^*(z) > 1 - \varepsilon\}$$

denotes the weak* slice of B_{X^*} determined by $z \in S_X$ and $\varepsilon > 0$. For an $x^* \in X^*$ and a $y \in Y$ the symbol $x^* \otimes y$ stands for the operator which acts from X into Y as follows: $(x^* \otimes y)(x) = x^*(x)y$.

Finally, let us cite a fact that we frequently use in the sequel.

Theorem 1.2 ([3], Theorem 2.1)**.** *For an operator $G \colon X \to Y$ with $\|G\| = 1$ the following assertions are equivalent:*

(i) *G is a Daugavet center.*

(ii) *For every $y_0 \in S_Y$, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$ there is an $x \in S(B_X, x_0^*, \varepsilon)$ with $\|Gx + y_0\| > 2 - \varepsilon$.*

(iii) *For every $y_0 \in S_Y$, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$ there is a $y^* \in S(B_{Y^*}, y_0, \varepsilon)$ with $\|G^* y^* + x_0^*\| > 2 - \varepsilon$.*

2. Banach spaces denying the Daugavet property

Definition 2.1. *We say that X denies the Daugavet property with a set $A \subset S_X$ if there is an $\varepsilon > 0$ such that for every $y \in A$ there exists an $x^* \in S_{X^*}$ satisfying*

$$\|\mathrm{Id} + x^* \otimes y\| < 2 - \varepsilon. \tag{2.1}$$

Definition 2.2. *We say that X star-denies the Daugavet property with a set $A \subset S_{X^*}$ if there is an $\varepsilon > 0$ such that for every $x^* \in A$ there exists a $y \in S_X$ satisfying (2.1).*

Lemma 2.3. *For $A \subset S_X$ the following assertions are equivalent:*

(i) *X denies the Daugavet property with A.*

(ii) *There is an $\varepsilon > 0$ such that for every $y \in A$ a functional $x^* \in S_{X^*}$ may be chosen so that every $x \in S(B_X, x^*, \varepsilon)$ fulfills $\|x+y\| < 2-\varepsilon$.*

(iii) *There is an $\varepsilon > 0$ such that for every $y \in A$ a functional $x^* \in S_{X^*}$ may be chosen so that every $y^* \in S(B_{X^*}, y, \varepsilon)$ fulfills $\|x^* + y^*\| < 2 - \varepsilon$.*

Proof. $(i) \Rightarrow (ii)$ We have that there is an $\varepsilon > 0$ such that for every $y \in A$ there exists an $x^* \in S_{X^*}$ satisfying

$$\|\mathrm{Id} + x^* \otimes y\| = \sup_{x \in B_X} \|x + x^*(x)y\| < 2 - \varepsilon.$$

Hence $\|x + x^*(x)y\| < 2 - \varepsilon$ for every $x \in B_X$. Let $x \in S(B_X, x^*, \varepsilon/2)$ then

$$\|x+y\| \leq \|x+x^*(x)y\|+\|y-x^*(x)y\| < 2-\varepsilon+|1-x^*(x)|\cdot\|y\| < 2-\varepsilon+\frac{\varepsilon}{2} = 2-\frac{\varepsilon}{2}$$

which implies (ii).

$(ii) \Rightarrow (i)$ Let ε and x^* be from (ii). It is sufficient to show that $\|x + x^*(x)y\| \leq 2 - \varepsilon/2$ for every $x \in B_X$. Let $x \in S(B_X, x^*, \varepsilon/2)$ then

$$\|x + x^*(x)y\| \leq \|x + y\| + \|y - x^*(x)y\| < 2 - \varepsilon + |1 - x^*(x)| \cdot \|y\| < 2 - \frac{\varepsilon}{2}.$$

Let $x \in S(B_X, -x^*, \varepsilon/2)$ then $-x \in S(B_X, x^*, \varepsilon/2)$. Hence $\|x - y\| < 2 - \varepsilon$ and

$$\|x + x^*(x)y\| \leq \|x - y\| + \|y + x^*(x)y\| < 2 - \varepsilon + |1 + x^*(x)| \cdot \|y\| < 2 - \frac{\varepsilon}{2}.$$

Finally, let $x \in B_X \setminus \big(S(B_X, x^*, \varepsilon/2) \cup S(B_X, -x^*, \varepsilon/2)\big)$ then

$$\|x + x^*(x)y\| \leq \|x\| + |x^*(x)| \cdot \|y\| \leq 2 - \frac{\varepsilon}{2}.$$

The equivalence $(i) \Leftrightarrow (iii)$ can be proved in a very similar fashion to $(i) \Leftrightarrow (ii)$ using the fact that the norms of an operator and of its adjoint coincide. □

Lemma 2.4. *For $A \subset S_{X^*}$ the following assertions are equivalent:*

(i) *X star-denies the Daugavet property with A.*

(ii) *There is an $\varepsilon > 0$ such that for every $x^* \in A$ a vector $y \in S_X$ may be chosen so that every $x \in S(B_X, x^*, \varepsilon)$ fulfills $\|x + y\| < 2 - \varepsilon$.*

(iii) *There is an $\varepsilon > 0$ such that for every $x^* \in A$ a vector $y \in S_X$ may be chosen so that every $y^* \in S(B_{X^*}, y, \varepsilon)$ fulfills $\|x^* + y^*\| < 2 - \varepsilon$.*

One can prove Lemma 2.4 the same way as Lemma 2.3. The following two lemmas form the main result of this section.

Lemma 2.5. *Let there exist $\delta > 0$ and $z^* \in S_{X^*}$ such that X denies the Daugavet property with $S(B_X, z^*, \delta) \cap S_X$. Then X is not a Daugavet domain.*

Proof. According to Definition 1.1 we must prove that any $G\colon X \to Y$ is not a Daugavet center for any Y. It is easy to see that if G is a Daugavet center then $G/\|G\|$ is as well, so we consider only the case $\|G\| = 1$.

Take the ε from item (ii) of Lemma 2.3. At first we show that if every $z \in S(B_X, z^*, \delta)$ satisfies $\|Gz\| \leq 1 - \varepsilon/2$ then G is not a Daugavet center.

Put $\varepsilon_0 := \min\{\varepsilon/2, \delta\}$, then for every $y \in S_Y$ and every $z \in S(B_X, z^*, \varepsilon_0)$ we have

$$\|y + Gz\| \leq 1 + \|Gz\| \leq 2 - \frac{\varepsilon}{2} \leq 2 - \varepsilon_0.$$

Theorem 1.2, item (ii) implies that G is not a Daugavet center.

So, we suppose that there is a $z_0 \in S(B_X, z^*, \delta)$ with

$$\|Gz_0\| > 1 - \frac{\varepsilon}{2}. \tag{2.2}$$

We can assume $\|z_0\| = 1$, because if $z_0 \in B_X$ fulfills $z^*(z_0) > 1 - \delta$ and (2.2) then $z_0/\|z_0\|$ does as well. In addition, (2.2) implies that there is a $y_0 \in S_Y$ with $\|y_0 - Gz_0\| < \varepsilon/2$. Since X denies the Daugavet property with $S(B_X, z^*, \delta) \cap S_X$, there is an $x^* \in S_{X^*}$ such that every $x \in S(B_X, x^*, \varepsilon)$ satisfies $\|x + z_0\| < 2 - \varepsilon$. Hence for every $x \in S(B_X, x^*, \varepsilon)$ we have

$$\|y_0 + Gx\| \leq \|y_0 - Gz_0\| + \|Gx + Gz_0\| < \frac{\varepsilon}{2} + \|x + z_0\| < 2 - \frac{\varepsilon}{2}.$$

By Theorem 1.2, item (ii) G is not a Daugavet center. □

Lemma 2.6. *Let there exist $\delta > 0$ and $z \in S_X$ such that X star-denies the Daugavet property with $S(B_{X^*}, z, \delta) \cap S_{X^*}$. Then X is not a Daugavet range.*

Using item (iii) of Lemma 2.4 and item (iii) of Theorem 1.2 one can prove Lemma 2.6 in a very similar fashion to Lemma 2.5.

3. Two-dimensional lattices denying the positive Daugavet property

Definition 3.1. *We say that F denies the positive Daugavet property with $A \subset S_F^+$ if there is an $\varepsilon > 0$ such that for every $a \in A$ there exists an $f^* \in S_{F^*}^+$ satisfying*

$$\|\mathrm{Id} + f^* \otimes a\| < 2 - \varepsilon. \tag{3.1}$$

Definition 3.2. *We say that F star-denies the positive Daugavet property with $A \subset S_{F^*}^+$ if there is an $\varepsilon > 0$ such that for every $f^* \in A$ there exists an $a \in S_F^+$ satisfying (3.1).*

The following two lemmas are complete analogs of Lemmas 2.3 and 2.4, so we skip their proofs.

Lemma 3.3. *For $A \subset S_F^+$ the following assertions are equivalent:*

(i) *F denies the positive Daugavet property with A.*

(ii) *There is an $\varepsilon > 0$ such that for every $a \in A$ a functional $f^* \in S_{F^*}^+$ may be chosen so that every $b \in S(B_F, f^*, \varepsilon) \cap B_F^+$ fulfills $\|a + b\| < 2 - \varepsilon$.*

(iii) *There is an $\varepsilon > 0$ such that for every $a \in A$ a functional $f^* \in S_{F^*}^+$ may be chosen so that every $g^* \in S(B_{F^*}, a, \varepsilon) \cap B_{F^*}^+$ fulfills $\|f^* + g^*\| < 2 - \varepsilon$.*

Lemma 3.4. *For $A \subset S^+_{F^*}$ the following assertions are equivalent:*

(i) *F star-denies the positive Daugavet property with A.*

(ii) *There is an $\varepsilon > 0$ such that for every $f^* \in A$ a vector $a \in S^+_F$ may be chosen so that every $b \in S(B_F, f^*, \varepsilon) \cap B^+_F$ fulfills $\|a+b\| < 2-\varepsilon$.*

(iii) *There is an $\varepsilon > 0$ such that for every $f^* \in A$ a vector $a \in S^+_F$ may be chosen so that every $g^* \in S(B_{F^*}, a, \varepsilon) \cap B^+_{F^*}$ fulfills $\|f^*+g^*\| < 2-\varepsilon$.*

Recall that $F^* = \mathbb{R}^2$ with the norm

$$\|(f_1, f_2)\|_{F^*} := \max_{(a_1,a_2)\in B_F} |f_1 a_1 + f_2 a_2|$$

and $F^{**} = F$. We introduce an order on F^* the same way as on F. It is easy to see that $\|(1,0)\|_{F^*} = \|(0,1)\|_{F^*} = 1$ and $\|(f_1, f_2)\|_{F^*} = \|(|f_1|, |f_2|)\|_{F^*}$ for every $(f_1, f_2) \in F^*$. Hence F^* is a two-dimensional lattice as well. Lemmas 3.3 and 3.4 evidently imply the following fact (which one can easily deduce from Definitions 3.1 and 3.2 as well).

Lemma 3.5. *Let $A \subset S^+_F$ and $\tilde{A} \subset S^+_{F^*}$.*

(a) *If F denies the positive Daugavet property with A then F^* star-denies the positive Daugavet property with A.*

(b) *If F star-denies the positive Daugavet property with $\tilde{A}$ then F^* denies the positive Daugavet property with $\tilde{A}$.*

Here is the key lemma of this section. In its proof we use the idea from Theorem 5.1 of [1].

Lemma 3.6. *Let there exist $w^* \in S^+_{F^*}$ and $\delta > 0$ such that F denies the positive Daugavet property with $S(B_F, w^*, \delta) \cap S^+_F$. Then $X_1 \oplus_F X_2$ is not a Daugavet domain for any X_1 and X_2.*

Proof. It is easy to see that $(X_1 \oplus_F X_2)^* = X_1^* \oplus_{F^*} X_2^*$ for every X_1 and X_2. Pick a $z^* = (z_1^*, z_2^*) \in S_{(X_1\oplus_F X_2)^*}$ with $(\|z_1^*\|, \|z_2^*\|) = w^*$. Then for a $y = (y_1, y_2) \in S(B_{X_1\oplus_F X_2}, z^*, \delta) \cap S_{X_1\oplus_F X_2}$ we have

$$\|z_1^*\|\|y_1\| + \|z_2^*\|\|y_2\| \geq z_1^*(y_1) + z_2^*(y_2) = z^*(y) > 1 - \delta.$$

Hence $a := (\|y_1\|, \|y_2\|) \in S(B_F, w^*, \delta) \cap S^+_F$. By item (ii) of Lemma 3.3 there exist $\varepsilon > 0$ and $f^* \in S^+_{F^*}$ such that every $b \in S(B_F, f^*, \varepsilon) \cap B^+_F$ satisfies $\|a + b\| < 2 - \varepsilon$.

Pick an $x^* = (x_1^*, x_2^*) \in S_{(X_1\oplus_F X_2)^*}$ with $(\|x_1^*\|, \|x_2^*\|) = f^*$. Then for every $x = (x_1, x_2) \in S(B_{X_1\oplus_F X_2}, x^*, \varepsilon)$ we have $b_x := (\|x_1\|, \|x_2\|) \in S(B_F, f^*, \varepsilon) \cap B^+_F$ and therefore

$$\begin{aligned} \|x + y\| &= \|(\|x_1 + y_1\|, \|x_2 + y_2\|)\| \\ &\leq \|(\|x_1\| + \|y_1\|, \|x_2\| + \|y_2\|)\| = \|a + b_x\| < 2 - \varepsilon. \end{aligned}$$

By item (ii) of Lemma 2.3 $X_1 \oplus_F X_2$ denies the Daugavet property for $S(B_{X_1\oplus_F X_2}, z^*, \delta) \cap S_{X_1\oplus_F X_2}$. So, Lemma 2.5 implies that $X_1 \oplus_F X_2$ is not a Daugavet domain. $\square$

The same conclusions based on item (iii) of Lemma 3.4, item (iii) of Lemma 2.4, and Lemma 2.6 prove the following fact:

Lemma 3.7. *Let there exist $w \in S_F^+$ and $\delta > 0$ such that F star-denies the positive Daugavet property with $S(B_{F^*}, w, \delta) \cap S_{F^*}^+$. Then $X_1 \oplus_F X_2$ is not a Daugavet range for any X_1 and X_2.*

4. Sums of spaces which are not Daugavet ranges

In this section we find a large class of those F which star-deny the positive Daugavet property with some $S(B_{F^*}, w, \delta) \cap S_{F^*}^+$. Throughout this and the following sections $e_1 := (1,0) \in S_F^+$, $e_2 := (0,1) \in S_F^+$, and the symbol $[a,b]$ is reserved for the line segment with the end points in $a, b \in F$.

Lemma 4.1. *Let D be a closed subset of $S_{F^*}^+$. Suppose for every $f^* \in D$ there exists an $\varepsilon > 0$ such that the property $P(f^*, \varepsilon) := \{$there is an $a \in S_F^+$ such that every $b \in S(B_F, f^*, \varepsilon) \cap B_F^+$ satisfies $\|a+b\| < 2 - \varepsilon\}$ holds true. Then F star-denies the positive Daugavet property with D.*

Proof. Note that if $P(f^*, \varepsilon)$ holds true then $P(f^*, \varepsilon_1)$ holds for every ε_1: $0 < \varepsilon_1 < \varepsilon$. Our goal is to show that there exists a common $\varepsilon_{min} > 0$ such that $P(f^*, \varepsilon_{min})$ holds true for every $f^* \in D$.

Consider the function $u(f^*)$: $D \to (0,1)$, $u(f^*) = \sup\{\varepsilon > 0\colon P(f^*, \varepsilon) \textit{ holds true}\}$. Let us prove that $u(f^*)$ reaches its minimum value on D. Since D is compact, it is sufficient to show that $u(f^*)$ is lower semicontinuous, i.e. that the set $u^{-1}\big((x,1)\big)$ is open for every $x \in [0,1)$.

Let $f^* \in u^{-1}\big((x,1)\big)$. This means that $u(f^*) = \sup\{\varepsilon > 0\colon P(f^*, \varepsilon) \textit{ holds true}\} > x$. Hence there exist $\varepsilon_0 > x$ and $a \in S_F^+$ such that every $b \in S(B_F, f^*, \varepsilon_0) \cap B_F^+$ fulfills $\|a+b\| < 2 - \varepsilon_0$.

Take an ε_1: $x < \varepsilon_1 < \varepsilon_0$ and put $\delta := \varepsilon_0 - \varepsilon_1$. The set $D \cap B_{F^*}(f^*, \delta)$ is a relative neighborhood of f^* in D. Let us show that $D \cap B_{F^*}(f^*, \delta) \subset u^{-1}\big((x,1)\big)$.

Let $f_1^* \in D \cap B_{F^*}(f^*, \delta)$. Then every $b \in S(B_F, f_1^*, \varepsilon_1) \cap B_F^+$ fulfills

$$f^*(b) \geq f_1^*(b) - \delta > 1 - \varepsilon_1 - \delta = 1 - \varepsilon_0.$$

Thus $b \in S(B_F, f^*, \varepsilon_0) \cap B_F^+$, so we have

$$\|a+b\| < 2 - \varepsilon_0 < 2 - \varepsilon_1.$$

This means that $u(f_1^*) \geq \varepsilon_1 > x$ and $f_1^* \in u^{-1}\big((x,1)\big)$. Consequently, $u^{-1}\big((x,1)\big)$ is open and $u(f^*)$ is lower semicontinuous. Then there exists an $f_0^* \in D$ such that

$$u(f_0^*) = \min_{f^* \in D} u(f^*).$$

Take an ε_{min}: $0 < \varepsilon_{min} < u(f_0^*)$ then $P(f^*, \varepsilon_{min})$ holds true for every $f^* \in D$. $\square$

Lemma 4.2. *Let $a \in S_F^+$ and $f^* \in S_{F^*}^+$. Suppose for every $\varepsilon > 0$ there is a $b \in S(B_F, f^*, \varepsilon) \cap B_F^+$ with $\|a+b\| \geq 2-\varepsilon$. Then there exists a $b_0 \in S_F^+$ such that $f^*(b_0) = 1$ and $[a, b_0] \subset S_F^+$.*

Proof. Consider a vanishing sequence $\{\varepsilon_n\}_{n=1}^{\infty}$, $\varepsilon_n > 0$. For every $n \in \mathbb{N}$ there exists a $b_n \in B_F^+$ with $f^*(b_n) > 1 - \varepsilon_n$ and $\|a + b_n\| \geq 2 - \varepsilon_n$.

Since B_F^+ is a compact set, there exists a subsequence $\{b_{n_i}\}_{i=1}^{\infty}$ of $\{b_n\}_{n=1}^{\infty}$ that converges to some $b_0 \in B_F^+$. Then $f^*(b_0) = 1$ and $\|a + b_0\| = 2$ which implies $[a, b_0] \subset S_F^+$. □

Denote $\mathfrak{N}_3$ the class of those F whose S_F^+ is a polygon which consists of at most three edges.

Lemma 4.3. *Let $F \notin \mathfrak{N}_3$. Then F star-denies the positive Daugavet property with $S_{F^*}^+$.*

Proof. Assume to the contrary that there exists an $f^* \in S_{F^*}^+$ such that for every $\varepsilon > 0$ and $a_0 \in S_F^+$ there is a $b \in S(B_F, f^*, \varepsilon) \cap B_F^+$ with $\|a_0 + b\| \geq 2 - \varepsilon$.

Consider the set $\Delta := \{a \in S_F^+ : f^*(a) = 1\}$. It is easy to see that Δ is a segment or a point. Put $a_0 := e_1$. By Lemma 4.2 there exists a $b_0 \in \Delta$ such that $[b_0, e_1] \subset S_F^+$. If we put $a_0 := e_2$ we obtain a $b_1 \in \Delta$ with $[b_1, e_2] \subset S_F^+$. Then $F \in \mathfrak{N}_3$, because S_F^+ consists of at most three segments: $[b_0, e_1]$, Δ and $[b_1, e_2]$. This contradiction completes the proof. □

Lemma 4.4. *Let S_F^+ be a polygon which consists of exactly three edges. Then there exists a $w^* = (w_1, w_2) \in S_{F^*}^+$ with $w_1 < 1$ and $w_2 < 1$ such that F star-denies the positive Daugavet property with $S_{F^*}^+ \setminus \mathring{B}_{F^*}(w^*, \delta_0)$ for every $\delta_0 > 0$.*

Proof. Since S_F^+ consists of three edges, it has four vertexes. The points e_1 and e_2 are two of them, denote h_1 and h_2 the remaining ones in such a way that $[e_1, h_1] \cup [e_2, h_2] \subset S_F^+$. There is the unique $w^* = (w_1, w_2) \in S_{F^*}^+$ such that $\{a \in S_F^+ : w^*(a) = 1\} = [h_1, h_2]$. It is obvious that $w_1 < 1$ and $w_2 < 1$.

Consider a $\delta_0 > 0$ and an $f^* \in S_{F^*}^+ \setminus \mathring{B}_{F^*}(w^*, \delta_0)$. Denote $\Delta := \{a \in S_F^+ : f^*(a) = 1\}$, it is a segment or a point. Assume that for every $\varepsilon > 0$ and $a \in S_F^+$ there exists a $b \in S(B_F, f^*, \varepsilon) \cap B_F^+$ with $\|a + b\| \geq 2 - \varepsilon$. By Lemma 4.2 there are $b_1, b_2 \in \Delta$ such that $[b_1, e_1] \subset S_F^+$ and $[b_2, e_2] \subset S_F^+$. Hence $b_1 \in [e_1, h_1]$ and $b_2 \in [e_2, h_2]$. Since $[e_1, h_1] \cap [e_2, h_2] = \emptyset$ then $\Delta \not\subseteq [e_1, h_1]$, $\Delta \not\subseteq [e_2, h_2]$, and Δ is a segment. Consequently, $\Delta \subset [h_1, h_2]$. But then $w^* = f^*$, so we come to contradiction.

Thus for every $f^* \in S_{F^*}^+ \setminus \mathring{B}_{F^*}(w^*, \delta_0)$ there are $\varepsilon > 0$ and $a \in S_F^+$ such that every $b \in S(B_F, f^*, \varepsilon) \cap B_F^+$ satisfies $\|a + b\| < 2 - \varepsilon$. Since $S_{F^*}^+ \setminus \mathring{B}_{F^*}(w^*, \delta_0)$ is closed, Lemma 4.1 implies the needed result. □

Denote $\mathfrak{N}_2$ the class of those F whose S_F^+ is a polygon which consists of at most two edges.

Corollary 4.5. *Let $F \notin \mathfrak{N}_2$. Then there is a $\delta > 0$ such that F star-denies the positive Daugavet property with $S(B_{F^*}, e_1, \delta) \cap S^+_{F^*}$.*

Proof. If $F \notin \mathfrak{N}_3$ then by Lemma 4.3 the statement is proved.

If S^+_F is a polygon which consists of exactly three edges then by Lemma 4.4 there exists a $w^* = (w_1, w_2) \in S^+_{F^*}$ with $w_1 < 1$ such that F star-denies the positive Daugavet property with $S^+_{F^*} \setminus \mathring{B}_{F^*}(w^*, \delta_0)$ for every $\delta_0 > 0$. Pick a $\delta_0 > 0$ with $\delta_0 + w_1 < 1$ and a δ such that $0 < \delta < 1 - w_1 - \delta_0$. Then every $f^* \in \mathring{B}_{F^*}(w^*, \delta_0)$ satisfies

$$f^*(e_1) < w^*(e_1) + \delta_0 = w_1 + \delta_0 < 1 - \delta.$$

Hence $\mathring{B}_{F^*}(w^*, \delta_0) \cap S(B_{F^*}, e_1, \delta) = \emptyset$. Thus F star-denies the positive Daugavet property with $S(B_{F^*}, e_1, \delta) \cap S^+_{F^*}$. □

We obtain the following fact by the successive application of Corollary 4.5 and Lemma 3.7.

Corollary 4.6. *Let $F \notin \mathfrak{N}_2$. Then $X_1 \oplus_F X_2$ is not a Daugavet range for any X_1 and X_2.*

5. Sums of spaces which are not Daugavet domains

Lemma 5.1. *Let $F^* \notin \mathfrak{N}_2$. Then $X_1 \oplus_F X_2$ is not a Daugavet domain for any X_1 and X_2.*

Proof. By Corollary 4.5 there is a $\delta > 0$ such that F^* star-denies the positive Daugavet property with $S(B_{F^{**}}, e_1, \delta) \cap S^+_{F^{**}}$. Recall that $F^{**} = F$. Therefore it follows from Lemma 3.5 that F denies the positive Daugavet property with $S(B_F, e_1, \delta) \cap S^+_F$. Then Lemma 3.6 gives the needed result. □

We characterize the class of those F such that $S^+_{F^*}$ is a polygon with at most two edges, with the help of the following notation. Consider an F whose S^+_F is a polygon with n edges. Denote $\hat{x}_1 := \max_{(1,y)\in S^+_F} y$ and $\hat{x}_2 := \max_{(x,1)\in S^+_F} x$. We say that F belongs to $\mathcal{F}_{n-1,n}$ if $\hat{x}_1 > 0$ and $\hat{x}_2 > 0$. If only one of $\hat{x}_j$ equals zero, we say that $F \in \mathcal{F}_{n,n}$. And if both $\hat{x}_1 = \hat{x}_2 = 0$ then $F \in \mathcal{F}_{n+1,n}$ (see Figure 1).

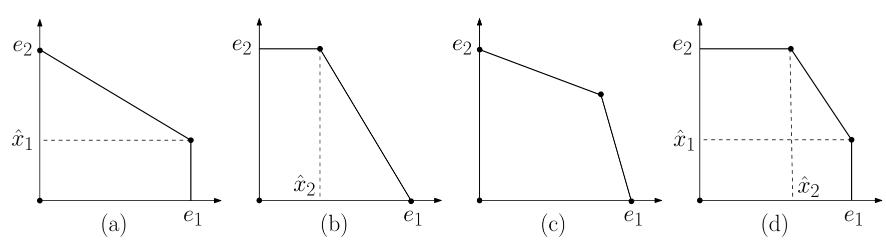

Figure 1. Those F whose S^+_F are presented on pictures (a) and (b), belong to $\mathcal{F}_{2,2}$. Picture (c) shows S^+_F of $F \in \mathcal{F}_{3,2}$ and (d) shows S^+_F of $F \in \mathcal{F}_{2,3}$.

Thus, $\mathfrak{N}_2 = \{\ell_1^{(2)}\} \cup \{\ell_\infty^{(2)}\} \cup \mathcal{F}_{2,2} \cup \mathcal{F}_{3,2}$. Let $n \in \mathbb{N}$ and $m \in \{n-1, n, n+1\}$. It is easy to see that $F^* \in \mathcal{F}_{n,m}$ if and only if $F \in \mathcal{F}_{m,n}$. Therefore, if $F^* \in \mathcal{F}_{2,2} \cup \mathcal{F}_{3,2}$ then $F \in \mathcal{F}_{2,2} \cup \mathcal{F}_{2,3}$. So, we obtain the following fact:

Corollary 5.2. *Let $F \notin \{\ell_1^{(2)}\} \cup \{\ell_\infty^{(2)}\} \cup \mathcal{F}_{2,2} \cup \mathcal{F}_{2,3} =: \mathfrak{M}_2$. Then $X_1 \oplus_F X_2$ is not a Daugavet domain for any X_1 and X_2.*

6. Examples of Daugavet centers acting from and into a sum of two Banach spaces

In this section we show that for every $F \in \mathfrak{M}_2$ there exists a Daugavet domain $X_1 \oplus_F X_2$, and for every $F \in \mathfrak{N}_2$ there is a Daugavet range $X_1 \oplus_F X_2$.

For $F = \ell_1^{(2)}$ and $F = \ell_\infty^{(2)}$ several examples of $X_1 \oplus_F X_2$ which are Daugavet domains and Daugavet ranges, are known. For instance, if X is a Daugavet domain then for every E the sum $X \oplus_\infty E$ is as well; and if X is a Daugavet range then $X \oplus_1 E$ is. If $G_1 \colon X_1 \to Y_1$ and $G_2 \colon X_2 \to Y_2$ are Daugavet centers then $G \colon X_1 \oplus_1 X_2 \to Y_1 \oplus_1 Y_2$ and $\tilde{G} \colon X_1 \oplus_\infty X_2 \to Y_1 \oplus_\infty Y_2$ which map every (x_1, x_2) into $(G_1 x_1, G_2 x_2)$, are Daugavet centers as well [3].

For future reference we mention the following fact:

Lemma 6.1 ([6], Lemma 2.8)**.** *If X has the Daugavet property then for every finite-dimensional subspace Y_0 of X, every $\varepsilon > 0$, and every slice $S(B_X, x^*, \varepsilon)$ there is an $x \in S(B_X, x^*, \varepsilon)$ such that every $y \in Y_0$ and $t \in \mathbb{R}$ fulfill*

$$\|y + tx\| \geq (1 - \varepsilon)(\|y\| + |t|).$$

Consider an $F \in \mathcal{F}_{2,2} \cup \mathcal{F}_{2,3}$. Denote $c_1 := (1, \hat{x}_1) \in S_F^+$ and $c_2 := (\hat{x}_2, 1) \in S_F^+$. Then $[c_1, c_2] \subset S_F^+$ (see Figure 2).

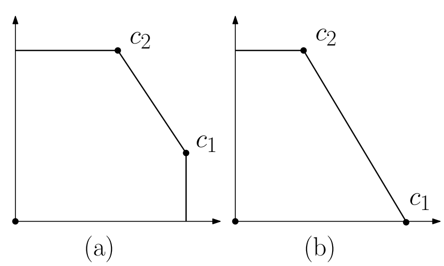

Figure 2. Picture (a) shows S_F^+ of $F \in \mathcal{F}_{2,3}$ and (b) presents S_F^+ of $F \in \mathcal{F}_{2,2}$.

Let the line containing $[c_1, c_2]$ have the equation $f_1 a_1 + f_2 a_2 = 1$ with $(f_1, f_2) \in S_{F^*}^+$. Remark that for every $w^* \in S_{F^*}^+$ and $\varepsilon > 0$ we have $S(B_F, w^*, \varepsilon) \cap [c_1, c_2] \neq \emptyset$. In other words, there exists an $(a_1, a_2) \in S(B_F, w^*, \varepsilon)$ such that $f_1 a_1 + f_2 a_2 = 1$.

Proposition 6.2. *Let X have the Daugavet property, $F \in \mathcal{F}_{2,2} \cup \mathcal{F}_{2,3}$, and let $(f_1, f_2) \in S^+_{F^*}$ be the functional described above. Then $G\colon X \oplus_F X \to X$, $G(x_1, x_2) = f_1 x_1 + f_2 x_2$ is a Daugavet center.*

Proof. At first, calculate $\|G\|$:

$$\|G\| = \sup_{(x_1,x_2)\in S_{X\oplus_F X}} \|f_1x_1 + f_2x_2\| = \sup_{(x_1,x_2)\in S_{X\oplus_F X}} (f_1\|x_1\| + f_2\|x_2\|) = 1.$$

Let $\varepsilon > 0$, $y_0 \in S_X$ and $x^* = (x_1^*, x_2^*) \in S_{(X\oplus_F X)^*}$.

By Lemma 6.1 there exists an $\tilde{x}_1 \in B_X$ with $x_1^*(\tilde{x}_1) \geq \|x_1^*\|(1 - \varepsilon/4)$ and

$$\|y_0 + t\tilde{x}_1\| \geq \left(1 - \frac{\varepsilon}{4}\right)(1 + |t|) \tag{6.1}$$

for every $t \in \mathbb{R}$. Using again Lemma 6.1 we have an $\tilde{x}_2 \in B_X$ with $x_2^*(\tilde{x}_2) \geq \|x_2^*\|(1 - \varepsilon/4)$ and

$$\|y + t\tilde{x}_2\| \geq \left(1 - \frac{\varepsilon}{4}\right)(\|y\| + |t|) \tag{6.2}$$

for every $y \in \operatorname{lin}\{y_0, \tilde{x}_1\}$ and every $t \in \mathbb{R}$.

Denote $w^* := (\|x_1^*\|, \|x_2^*\|) \in S^+_{F^*}$. Let $(a_1, a_2) \in S(B_F, w^*, 3\varepsilon/4)$ such that $f_1a_1 + f_2a_2 = 1$. Then for $x := (a_1\tilde{x}_1, a_2\tilde{x}_2) \in B_{X\oplus_F X}$ we have

$$\begin{aligned} x^*(x) &= a_1x_1^*(\tilde{x}_1) + a_2x_2^*(\tilde{x}_2) \geq \left(1 - \frac{\varepsilon}{4}\right)(a_1\|x_1^*\| + a_2\|x_2^*\|) \\ &\geq \left(1 - \frac{\varepsilon}{4}\right)\left(1 - \frac{3\varepsilon}{4}\right) > 1 - \varepsilon. \end{aligned}$$

Hence $x \in S(B_{X\oplus_F X}, x^*, \varepsilon)$ and

$$\|y_0 + Gx\| = \|y_0 + f_1a_1\tilde{x}_1 + f_2a_2\tilde{x}_2\|$$

by (6.2)

$$> \left(1 - \frac{\varepsilon}{4}\right)(\|y_0 + f_1a_1\tilde{x}_1\| + f_2a_2)$$

by (6.1)

$$> \left(1 - \frac{\varepsilon}{4}\right)^2(1 + f_1a_1 + f_2a_2) = 2\left(1 - \frac{\varepsilon}{4}\right)^2 > 2 - \varepsilon.$$

Theorem 1.2, item (ii) implies that G is a Daugavet center. □

Corollary 6.3. *For an F there exists a Daugavet domain $X_1 \oplus_F X_2$ if and only if $F \in \mathfrak{M}_2$.*

Remark 6.4. *Note that $\mathfrak{M}_2 \not\subseteq \mathfrak{N}_2$. Then Corollary 6.3 and Corollary 4.6 imply that there exist Daugavet domains which are not Daugavet ranges.*

Now we present more examples of Daugavet centers acting from $X_1 \oplus_F X_2$ where $F = \ell_1^{(2)}$ or $F = \ell_\infty^{(2)}$.

Proposition 6.5. *Let X have the Daugavet property. Then*

(a) *The operator $G\colon X \oplus_1 X \to X$, $G(x_1, x_2) = x_1 + x_2$ is a Daugavet center.*

(b) *For every* $f_1, f_2 > 0$ *the operator* $G\colon X \oplus_\infty X \to X$, $G(x_1, x_2) = f_1 x_1 + f_2 x_2$ *is a Daugavet center.*

Proposition 6.5 can be proved the same way as Proposition 6.2.

Proposition 6.6. *Let* X *have the Daugavet property,* $F \in \mathcal{F}_{2,2} \cup \mathcal{F}_{3,2}$, *and let* $(f_1, f_2) \in S_F^+$ *be the vector described above. Then* $G\colon X \to X \oplus_F X$, $Gx = (f_1 x, f_2 x)$ *is a Daugavet center.*

Proof. Consider the adjoint operator $G^*\colon X^* \oplus_{F^*} X^* \to X^*$. For every $(x_1^*, x_2^*) \in X^* \oplus_{F^*} X^*$ and every $x \in X$ we have

$$G^*(x_1^*, x_2^*)(x) = \langle (f_1 x, f_2 x), (x_1^*, x_2^*) \rangle = f_1 x_1^*(x) + f_2 x_2^*(x).$$

Consequently, $G^*(x_1^*, x_2^*) = f_1 x_1^* + f_2 x_2^*$ for every $(x_1^*, x_2^*) \in X^* \oplus_{F^*} X^*$. By Proposition 6.2 G^* is a Daugavet center. The equation (1.1) implies that if G^* is a Daugavet center then G is as well. □

Corollary 6.7. *For an* F *there exists a Daugavet range* $X_1 \oplus_F X_2$ *if and only if* $F \in \mathfrak{N}_2$.

Remark 6.8. *Since* $\mathfrak{N}_2 \not\subseteq \mathfrak{M}_2$, *we have the examples of Daugavet ranges which are not Daugavet domains.*

Proposition 6.9 which gives more examples of Daugavet centers acting into $X_1 \oplus_F X_2$ for $F = \ell_1^{(2)}$ and $F = \ell_\infty^{(2)}$, can be proved in a very similar way to Proposition 6.6.

Proposition 6.9. *Let* X *have the Daugavet property. Then*

(a) *The operator* $G\colon X \to X \oplus_\infty X$, $Gx = (x, x)$ *is a Daugavet center.*

(b) *For every* $f_1, f_2 > 0$ *the operator* $G\colon X \to X \oplus_1 X$, $Gx = (f_1 x, f_2 x)$ *is a Daugavet center.*

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References

[1] Bilik D., Kadets V., Shvidkoy R., Werner D., Narrow operators and the Daugavet property for ultraproducts, Positivity, 2005, 9, 45–62

[2] Bourgain J., Rosenthal H.P., Martingales valued in certain subspaces of L^1, Israel J. Math., 1980, 37, 54–75

[3] Bosenko T., Kadets V., Daugavet centers, Zh. Mat. Fiz. Anal. Geom., (in press), preprint available at http://arxiv.org/abs/0910.4503

[4] Daugavet I.K., On a property of completely continuous operators in the space C, Uspekhi Mat. Nauk, 1963, 18, 157–158, (in Russian)

[5] Kadets V.M., Shepelska V., Werner D., Quotients of Banach spaces with the Daugavet property, Bull. Pol. Acad. Sci. Math., 2008, 56, 131-147

[6] Kadets V.M., Shvidkoy R.V., Sirotkin G.G., Werner D., Banach spaces with the Daugavet property, Trans. Amer. Math. Soc., 2000, 352, 855–873

[7] Kadets V.M., Werner D., A Banach space with the Schur and the Daugavet property, Proc. Amer. Math. Soc., 2004, 132, 1765–1773

[8] Lindenstrauss J., Tzafriri L., Classical Banach Spaces II: Function spaces, Springer-Verlag, Berlin-Heidelberg-New York, 1979

[9] Lozanovskii G.Ya., On almost integral operators in KB-spaces, Vestnik Leningrad Univ. Mat. Mekh. Astr., 1966, 21, 35–44, (in Russian)

[10] Popov M.M., Daugavet type inequalities for narrow operators in the space L_1, Mat. Stud., 2003, 20, 75–84

[11] Shvidkoy R.V., Geometric aspects of the Daugavet property, J. Funct. Anal., 2000, 176, 198–212

[12] Talagrand M., The three-space problem for L^1, J. Amer. Math. Soc., 1990, 3, 9–29

[13] Werner D., Recent progress on the Daugavet property, Irish Math. Soc. Bull., 2001, 46, 77–97

[14] Werner D., The Daugavet equation for operators on function spaces, J. Funct. Anal., 1997, 143, 117–128

[15] Wojtaszczyk P., Some remarks on the Daugavet equation, Proc. Amer. Math. Soc., 1992, 115, 1047–1052

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